

A. Kievsky, M. Viviani, R. Álvarez-Rodríguez, M. Gattobigio,
A. Deltuva

Universal behavior of few-boson systems using potential models

Received: date / Accepted: date

Abstract The universal behavior of a three-boson system close to the unitary limit is encoded in a simple dependence of many observables in terms of few parameters. For example the product of the three-body parameter κ_* and the two-body scattering length a , $\kappa_* a$ depends on the angle ξ defined by $E_3/E_2 = \tan^2 \xi$. A similar dependence is observed in the ratio a_{AD}/a with a_{AD} the boson-dimer scattering length. We use a two-parameter potential to determine this simple behavior and, as an application, to compute a_{AD} for the case of three ^4He atoms.

Keywords few-body systems · universal behavior · Efimov physics

1 Introduction

The study of few-boson systems close to the unitary limit is an intense subject of research nowadays. For identical bosons, the unitary limit is defined when the two-body scattering length $a \rightarrow \infty$. In this limit the three-boson system shows the Efimov effect [1; 2]. Moreover, close to the unitary limit the system manifests universal behavior: the details of the two-body interaction are not important and its spectrum is determined essentially by a and the three-body parameter κ_* which defines the energy $E_* = \hbar^2 \kappa_*^2 / m$ of level n_* at the unitary limit, here m is the boson mass (for a recent review see Ref. [3]). At the unitary limit the spectrum shows a discrete scaling invariance (DSI): an infinite series of bound states appears distributed geometrically and accumulates at zero energy. The ratio of two consecutive energy states is constant, $E_3^n / E_3^{n+1} = e^{2\pi/s_0}$, with the universal number $s_0 \approx 1.00624$.

The universal characteristics of the system can be exploited studying the dynamics using potential models constructed in such a way that the control parameters of the spectrum are reproduced. For example a two-parameter potential as a gaussian can be used to this aim [4; 5]. Universal behavior manifests in a simple dependence of many observables on the angle defined by the ratio $E_3/E_2 = \tan^2 \xi$. This is the case of the product $\kappa_* a$ and, in the case in which the two bosons forms a dimer, the ratio a_{AD}/a between the boson-dimer

A. Kievsky
Istituto Nazionale di Fisica Nucleare, Largo Pontecorvo 3, 56127 Pisa, Italy
E-mail: kievsky@pi.infn.it

M. Viviani
Istituto Nazionale di Fisica Nucleare, Largo Pontecorvo 3, 56127 Pisa, Italy

R. Álvarez-Rodríguez
Escuela Técnica Superior de Arquitectura, Universidad Politécnica de Madrid, Avda. Juan Herrera 4, E-28040 Madrid, Spain

M. Gattobigio
Université de Nice-Sophia Antipolis, Institut Non-Linéaire de Nice, CNRS, 1361 route des Lucioles, 06560 Valbonne, France

A. Deltuva
Institute of Theoretical Physics and Astronomy, Vilnius University, A. Goštauto St. 12, LT-01108 Vilnius, Lithuania

scattering length a_{AD} and the two-body scattering length a . These relations are exactly fulfilled in the zero-range limit and, as we will show, range corrections can be introduced using the potential models. In fact the gaussian potentials can be used to determine the dependence on the angle ξ of the observables defined above. The first one, $\kappa_* a$, is used to define the gaussian level function [4]. Moreover a gaussian potential verifying $E_3/E_2 = \tan^2 \xi$ can be used to compute the ratio a_{AD}/a at that angle. After introducing range corrections we compute this ratio and, as an application, we determine a_{AD} for the case of a system composed by three ^4He atoms.

2 The gaussian level function

In the case of a zero-range interaction the $L = 0$ spectrum of three bosons is determined by the Efimov radial law

$$E_3^n / (\hbar^2 / ma^2) = \tan^2 \xi_n, \quad (1)$$

$$\kappa_* a = e^{(n-n^*)\pi/s_0} \frac{e^{-\Delta(\xi)/2s_0}}{\cos \xi}. \quad (2)$$

where a is the two-body scattering length and E_3^n is the energy of level n . The binding momentum κ_* gives the energy of the system at the unitary limit. The spectrum is determined by the knowledge of the universal function $\Delta(\xi)$ which is equal for all levels (a parametrization of the universal function is given in [3]). In fact knowing one energy value the complete spectrum is determined. The zero-range theory is not always sufficiently accurate to describe real systems and range corrections have to be introduced at some level. Since close to the unitary limit the details of the interaction are not important, it is possible to construct a potential model in order to capture the essential ingredients of the dynamics. Following this strategy a minimal information that preserves universal behavior is encoded in the effective range expansion for two particles, $k \cot \delta = -1/a + r_{\text{eff}} k^2/2$. With k the relative momentum and r_{eff} the effective range. At low energies this perturbative expansion is well fulfilled and, in the case of shallow states, it can be extended to negative energies relating a , r_{eff} and the two-body energy E_2 :

$$\frac{1}{a_B} = \frac{1}{a} + \frac{r_{\text{eff}}}{2a_B^2}, \quad (3)$$

where we have introduced the energy length from the relation $E_2 = \hbar^2 / ma_B^2$. Therefore a two-parameter potential describing E_2 and a will also describe r_{eff} . Accordingly we define a local and a non local gaussian potential

$$V^L(r) = -V_0^L e^{-r^2/r_0^2}, \quad (4)$$

$$V^{NL}(k, k') = -V_0^{NL} e^{-k^2/k_0^2} e^{-k'^2/k_0^2}, \quad (5)$$

with the strengths V_0^L , V_0^{NL} and the ranges r_0 , k_0^{-1} determined to describe particular values of a and E_2 of a two-boson system. If these values are experimental values we call this set of values a physical point. Once this point is fixed the strength of the potential can be varied to reach the unitary limit. With the potentials defined above the lengths, momenta and energies scale with r_0 , k_0 and \hbar^2 / mr_0^2 (or $\hbar^2 k_0^2 / m$), respectively. Therefore the local gaussian (LG) and the nonlocal gaussian (NLG) potentials define a particular path to the unitary limit that encompasses all the local and nonlocal gaussian potentials. In particular, for the ground state, the values of the effective range and strength at unitary are $r_{\text{eff}} = 1.43522 r_0$ and $\lambda V_0^L = 2.6840 \hbar^2 / mr_0^2$ (local gaussian) and $r_{\text{eff}} = 3.19154 / k_0$ and $\lambda V_0^{NL} = 0.126987 \hbar^2 k_0^2 / m$ (nonlocal gaussian).

The potentials defined in Eqs.(4,5) can be used to describe the three-boson system close to the unitary limit. The Efimov law of Eq.(2) suggests the following representation of the gaussian $L = 0$ spectrum of three-bosons

$$E_3^n / E_2 = \tan^2 \xi_n, \quad (6)$$

$$\kappa_*^n a_B = \frac{e^{-\tilde{\Delta}_n(\xi)/2s_0}}{\cos \xi}. \quad (7)$$

where κ_*^n defines the energy of level n at the unitary limit, $E_*^n = \hbar^2(\kappa_*^n)^2/m$, and the gaussian level function is defined as

$$\tilde{\Delta}_n = s_0 \ln \left(\frac{E_3^n + E_2}{E_*^n} \right). \quad (8)$$

The scaling properties of the gaussian potentials are such that the level function is the same for all local gaussian and for all nonlocal gaussian, being the local and nonlocal level functions slightly different for the ground state level $n = 0$. As $n > 0$ both level functions tend to be equal and tend to the zero-range function, $\tilde{\Delta}_n \rightarrow \Delta$. This behavior is shown in Fig.1 in which the LG and NLG level functions $\tilde{\Delta}_n(\xi)$ are shown for the ground and first excited state levels $n = 0, 1$. The universal zero-range function is shown as well and completely overlap with the NLG level function calculated for the third excited state $\tilde{\Delta}_3(\xi)$. In particular the ground state level function $\tilde{\Delta}_0(\xi)$ incorporate range corrections and can be used to determine the corresponding three-body parameter κ_*^0 [4; 5].

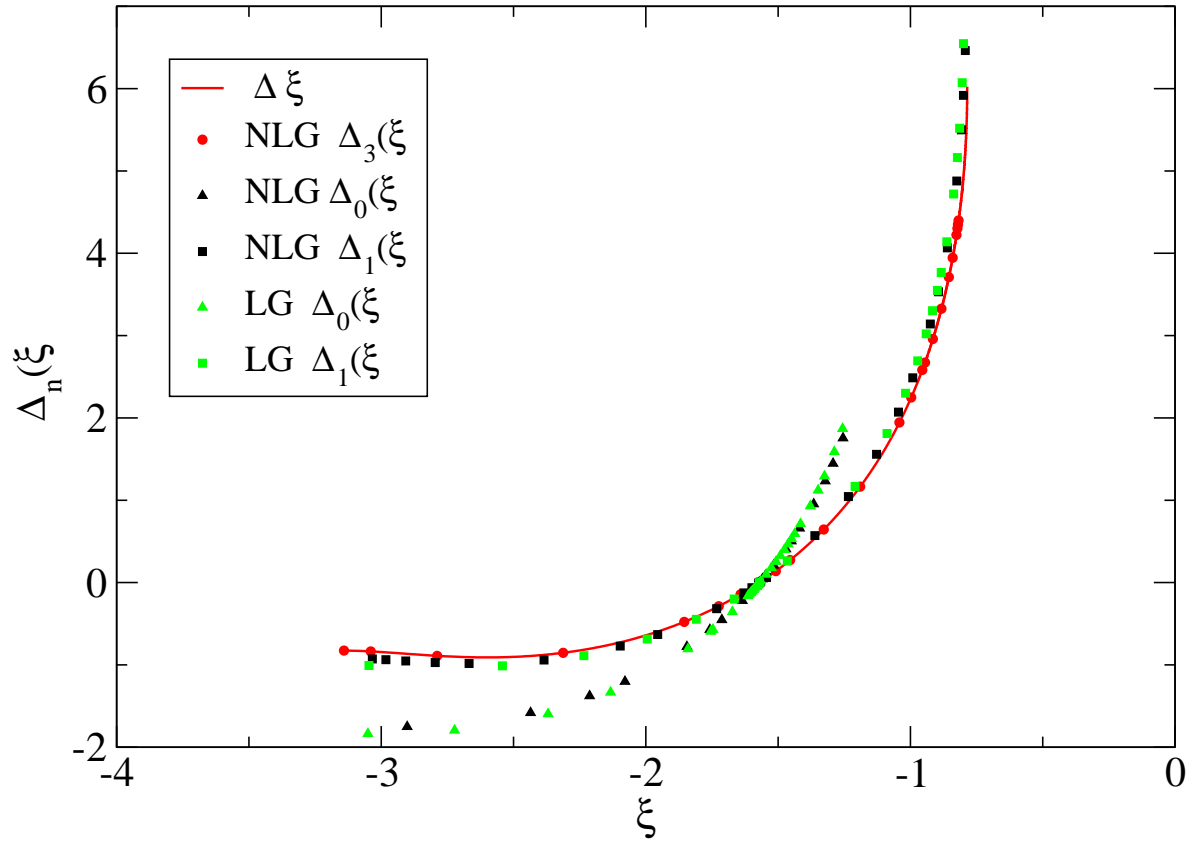


Fig. 1 The LG and NLG level functions $\tilde{\Delta}_n(\xi)$ are shown for the ground state and first excited state levels $n = 0, 1$. In the NLG case the third excited state ($n = 3$) level function $\tilde{\Delta}_3(\xi)$ (red circles) completely overlap with the zero-range universal function $\Delta(\xi)$ (red solid curve).

We now discuss the ξ dependence of the ratio a_{AD}/a . As discussed by Efimov [6] (see also Ref. [3]), in the zero-range limit, the ratio a_{AD}/a has the form

$$\frac{a_{AD}}{a} = d_1 + d_2 \tan[s_0 \ln(\kappa_* a) + d_3] \quad (9)$$

with d_1 , d_2 and d_3 universal numbers (numerical values are given in Refs. [3; 7]). Due to the ξ dependence of the product $\kappa_* a$, the above ratio depends on this angle too. Accordingly it is possible to study this ratio using potentials models. Following Ref. [7] we introduce range corrections by studying the ratio a_{AD}/a_B . We calculate it for different values of ξ and, in particular, we consider valid the following relation at the same ξ value:

$$\frac{a_{AD}}{a_B} = \frac{a_{AD}}{a_B} \Big|_{\text{gaussian}}, \quad (10)$$

where the first ratio refers to the values predicted by any realistic interaction or experimental values whereas *gaussian* refers to the values calculated with the LG or NLG potentials. As an application we calculate the value of the atom-dimer scattering length a_{AD} for a system composed by three ^4He atoms. This system is well described by the realistic LM2M2 potential of Aziz [8] for which $E_2 = 1.303$ mK. Using this potential the three-boson ground state energy is $E_3^0 = 126$ mK and, accordingly, the angle ξ is obtained from $E_3/E_2 = \tan^2 \xi = 97.0$. This ratio can be reproduced using a gaussian potential and with that potential the value of a_{AD} can be calculated using standard methods. After this straightforward procedure we obtain

$$a_{AD}|_{\text{LM2M2}} = a_B|_{\text{LM2M2}} \frac{a_{AD}}{a_B} \Big|_{\text{gaussian}} \approx 212a_0 \quad (11)$$

to be compared to $217 a_0$ obtained from a direct calculation using the LM2M2 interaction [9; 10]. As can we see the simple dependence of the ratio a_{AD}/a_B on the angle ξ is well fulfilled and, furthermore, the gaussian description reproduces this dependence with good approximation.

3 Conclusions

We have studied the three-boson system using potential models of the gaussian form. This simple representation of the interaction seems to capture the main aspects of the low energy dynamics close to the unitary limit. We have looked at the ratio E_3/E_2 defining the angle ξ . At the unitary limit it corresponds to $\xi = -\pi/2$. The DSI manifests at fixed values of ξ , this means that the infinite series of states observed at $\xi = -\pi/2$ can be observed also along the line in which ξ remains constant and the energy ratios along that line verify the same relation $E_3^n/E_3^{n+1} = e^{2\pi/s_0}$. It should be noticed that the case $\xi = -\pi/2$ corresponds to a single value of E_2 (equal to zero), whereas E_2 varies along the line in which ξ is constant and therefore the values of E_2 at which E_3^n and E_3^{n+1} are calculate are different. This property allows to study DSI at finite values of E_2 .

In the present work we have exploited the property that some observables close to the unitary limit have a simple dependence, they are functions of the angle ξ . First we have analysed the product $\kappa_*^n a_B$ from which we have defined the gaussian level function $\tilde{\Delta}_n(\xi)$. In second place we have analysed the ratio a_{AD}/a_B and the gaussian values for this ratio have been used to determine a_{AD} for the case of atom-dimer collision of three ^4He atoms at zero energy. From the results we observe that ξ dependence is well verified and, the gaussian model reproduces well the LM2M2 value for a_{AD} . Further investigations along this line are under way.

Acknowledgements This work was partly supported by Ministerio de Economía y Competitividad (Spain) under contracts MTM2015-63914-P and FPA2015-65035-P. Part of the calculations of this work were performed in the high capacity cluster for Physics, funded in part by Universidad Complutense de Madrid and in part with Feder funds as a contribution to the Campus of International Excellence of Moncloa, CEI Moncloa. R.A.R thanks Ministerio de Educación, Cultura y Deporte (Spain) for the José Castillejo fellowship in the framework of Plan Estatal de Investigación Científica y Técnica y de Innovación 2013-2016.

References

1. V. Efimov, Phys. Lett. B **33**, 563 (1970)
2. V. Efimov, Sov.J. Nucl. Phys. **12**, 589 (1971)
3. E. Braaten and H.-W. Hammer, Phys. Rep. **428**, 259 (2006)
4. A. Kievsky and M. Gattobigio, Phys. Rev. A **92**, 062715 (2015)
5. R. Álvarez-Rodríguez, A. Deltuva, M. Gattobigio, and M. Viviani, Phys. Rev. A **93**, 062701 (2016)
6. V. Efimov, Sov.J. Nucl. Phys. **29**, 546 (1979)
7. A. Kievsky and M. Gattobigio, Phys. Rev. A **87**, 052719 (2013)
8. R.A. Aziz and M.J. Slaman, J. Chem. Phys. **94**, 8047 (1991)
9. E.A. Kolganova, A.K. Motovilov, W. Sandhas, Phys. Part. Nucl. **40**, 206 (2009)
10. A. Kievsky, E. Garrido, C. Romero-Redondo and P. Barletta, Few-Body Syst., **51**, 259 (2011)

